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# N-SOFT TOPOLOGICAL STRUCTURES AND HOMEOMORPHISMS: A FRAMEWORK FOR UNCERTAINTY IN MULTI-CRITERIA SYSTEMS

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## ABSTRACT

This study investigates the topological structure of N-soft sets and introduces the notion of N-soft homeomorphism, which serves as a formal tool for preserving structural consistency in multi-criteria decision-making systems. Building upon the existing framework of N-soft topology, the study extends previously established notions such as continuity, interior, and closure to define topological equivalence within this context. We present a novel application of N-soft homeomorphism in the context of healthcare prioritization, using Behavioral Risk Factor Surveillance System data for US adults aged 35-44. By constructing N-soft topological spaces under both equal-weighted and priority-weighted scoring models for indicators such as fruit intake, vegetable intake, physical activity, and obesity, we demonstrate that the resulting decision spaces are N-soft homeomorphic. This topological equivalence ensures that consistent decision structures are maintained even as evaluation priorities shift. The findings underscore the robustness and flexibility of N-soft topology-based frameworks for dynamic decision-making environments.

**Keywords:** N-soft set, N-soft homeomorphism, Healthcare prioritization, Topological decision model, BRFSS, Multi-Criteria Decision-Making.

## 1 INTRODUCTION

In various applied disciplines such as economics, humanities, medical sciences, and environmental sciences decision-making often involves complex and context-dependent uncertainties that vary across individuals and systems. To address these uncertainties, researchers have proposed numerous new theoretical frameworks, including fuzzy set theory, rough set theory, intuitionistic fuzzy sets, and soft set theory. One of these is the fuzzy set theory



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introduced by Zadeh [1]. In this theory, Zadeh generalized the set of truth values to the interval of real numbers [0,1]. Thus, the degree of membership of an element to a set could be expressed with greater nuance. A fuzzy set is defined through its membership function [1].

Like other theories, the theory of fuzzy sets also has its challenges. According to Molodtsov [2], one major difficulty lies in the subjectivity and case-dependence of membership function construction. Since the membership degree of an element to a fuzzy set must be assigned individually, determining this degree often relies on expert judgment or arbitrary assumptions, which may vary significantly across applications. This subjectivity leads to inconsistency, lack of reproducibility, and ambiguity in practical implementations. Hence, there was a need for a set theory that could address uncertainty without depending on the explicit formulation of membership functions. In response, Molodtsov [2] proposed a novel approach to modeling uncertainty and instability by introducing soft set theory, which is free from these limitations. He successfully applied this theory to various fields, including the Riemann integral, Perron integral, and measure theory. This led to a growing interest in soft set theory across various fields of mathematics and applied sciences. Early research explored its foundational aspects, including algebraic structures, mappings, and topological frameworks. Subsequently, soft sets found wide-ranging applications in practical domains such as decision-making, medicine, and computer sciences [3-7].

The development of soft set theory has led to significant applications across many areas of mathematics. Ali et al. [8] investigated some algebraic operations on soft sets. Furthermore, by combining the concepts of soft sets and topology, Shabir and Naz [9] introduced the idea of soft topology, examining soft open sets, soft closed sets, and soft separation axioms in soft topological spaces. The concept of soft neighborhoods of points was later studied in detail by Nazmul and Samanta [27] in their work on neighborhood properties of soft topological spaces. Aktaş and Çağman [10] defined soft groups and they compared soft sets with fuzzy sets and rough sets. Additionally, Çağman et al. [11] explored the fundamental properties of soft sets and the concept of soft topology. Kharal and Ahmad [12] studied the images and preimages of soft sets under soft mapping. Later, Das and Samanta [13] introduced soft real numbers and investigated their properties. Aygünoğlu and Aygün [14] studied concepts such as soft compactness and product soft topology. Prior to practical implementations, several studies focused on soft separation axioms from a theoretical standpoint. For instance, Çağman et al. [11] discussed soft $T_0$, $T_1$, and $T_2$ separation properties, while Kharal and Ahmad [12] explored the role of soft mappings in separation. Building on these foundational results, Al-Shami [15]

provided the first practical application of soft separation axioms in a medical decision-making context. Subsequently, Al-Shami et al. [16] studied soft compactness and soft connectedness concepts through soft semi-open sets, which generalize soft open sets. Inspired by these works, many researchers have continued to conduct further studies on soft topology [17–20].

Although classical soft sets rely on binary evaluations, real-life decision-making often involves multi-level or graded assessments, such as rankings, scores, or star-based ratings. These non-binary structures cannot be captured adequately by standard soft sets, leading to limitations in modeling complex evaluation systems. To address this limitation, Fatimah et al. [21] proposed the concept of N-soft sets as an extended form of soft sets. Since the introduction of this concept, numerous studies have been conducted on N-soft sets. Riaz et al. [22] defined some new operations on N-soft sets and introduced the concept of N-soft topology. As a generalization of N-soft sets, Akram et al. [23] introduced hesitant N-soft sets and obtained various applications. Kamacı and Petchimuthu [24] proposed the concept of bipolar N-soft sets and established some decision-making approaches. Liu et al. [25] generalized both neutrosophic uncertain sets and N-soft sets by introducing neutrosophic uncertain N-soft sets.

In this paper, we introduce the concept of N-soft homeomorphism as a novel topological construct within the framework of N-soft sets and demonstrate its applicability to multi-criteria healthcare prioritization problems. We explore the structural properties and applications of N-soft sets, with a particular emphasis on their topological foundations. We formally define operations such as restricted intersection, union, and weak complement, along with key topological constructs such as open/closed sets, subspaces, interior, closure, and continuity. Furthermore, we investigate the concept of N-soft homeomorphism and its role in ensuring structural integrity between topological spaces. As a practical application, we utilize the framework of N-soft homeomorphism to analyze health prioritization systems. Using Behavioral Risk Factor Surveillance System (BRFSS) data [26] for U.S. adults aged 35–44, we demonstrate how structurally consistent decision models can be preserved across varying policy scenarios through topologically equivalent transformations.

## 2 PRELIMINARIES: N-SOFT SETS

**Definition 2.1.** [2] Let $U$ be a universal set and $E$ be a set of parameters. Given a subset $A \subseteq E$ and a mapping $F: A \to 2^U$, the pair $(F, A)$ is called a soft set over $U$.

**Definition 2.2.** [21] Let $U$ be a universal set of objects, $E$ a set of attributes, and $R = \{0,1,2,\dots,N-1\}$ a set of ordered ratings where $N \in \{2,3,4,\dots\}$. Suppose that $A \subseteq E$. An N-soft set over $U$ is defined as a mapping $F: A \to 2^{U\times R}$ where $F(a)(u) = r_a$, where for each $a \in A$ and $u \in U$, there exists exactly one pair $(u, r_a) \in U \times R$. In this framework, for every attribute $a$, each object $u \in U$ is assigned precisely one rating $r_a$ from $R$, representing its evaluation with respect to $a$.

Throughout this paper, unless otherwise specified, it is assumed that $U = \{u_i : i = 1,2,\dots,p\}$ and $A = \{a_j : j = 1,2,\dots,q\}$ are finite sets. An N-soft set can be conveniently represented in a tabular form, where each entry corresponds to $(u_i, r_{ij}) \in F(a_j)$, or equivalently as $F(a_j)(u_i) = r_{ij}$, as depicted in Table 1. Unlike classical soft sets where elements are simply associated with subsets, in N-soft sets each object is associated with a specific rating from an ordered set.

***Table 1. Tabular representation of an N-Soft Set***

| $(F, A, N)$ | $a_1$ | $a_2$ | $\cdots$ | $a_q$ |
|---|---|---|---|---|
| $u_1$ | $r_{11}$ | $r_{12}$ | $\cdots$ | $r_{1q}$ |
| $u_2$ | $r_{21}$ | $r_{22}$ | $\cdots$ | $r_{2q}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $u_p$ | $r_{p1}$ | $r_{p2}$ | $\cdots$ | $r_{pq}$ |

This tabular representation offers a systematic and convenient framework for the analysis and manipulation of N-soft sets in practical applications.

**Remark.** In application-specific contexts, we may denote $U$ as a set of items, and a subset $K \subseteq E$ as selected evaluation parameters. The mapping may then be written as:

$$T : K \to 2^{U\times R}, \quad T(k) = \{(u, l_k(u))\}$$

where $l_k(u) \in R$ represents the score of objects $u$ with respect to parameter $k$. In such a case, the N-soft set is denoted as $(T, K, N)$.

**Definition 2.4.** [21] Let $U$ be a universal set of objects. The restricted intersection of two N-soft sets $(F, A, N_1)$ and $(G, B, N_2)$ is defined as follows: For each $a \in A \cap B$ and $u \in U$, if $(u, r_a^1) \in F(a)$ and $(u, r_a^2) \in G(a)$, then $r_a = min\,(r_a^1, r_a^2)$ and $(u, r_a) \in H(a)$. Consequently, the restricted intersection of $(F, A, N_1)$ and $(G, B, N_2)$ is given by $(F, A, N_1) \cap_R (G, B, N_2) = (H, A \cap B, min(N_1, N_2))$.

**Definition 2.5.** [21] Let $U$ be a universal set of objects, and let $(F, A, N_1)$ and $(G, B, N_2)$ be N-soft sets. The restricted union of these sets is defined as follows: For each $a \in A \cap B$ and $u \in U$, if $(u, r_a^1) \in F(a)$ and $(u, r_a^2) \in G(a)$, then $r_a = max\,(r_a^1, r_a^2)$, and $(u, r_a) \in K(a)$. Hence, $(F, A, N_1) \cup_R (G, B, N_2) = (K, A \cap B, max(N_1, N_2))$.

**Example 2.6.** Let $R = \{r_1, r_2, r_3\}$ be a set of three residences and $K = \{k_1, k_2, k_3, k_4, k_5\}$ be a set of attributes, where: $k_1$: expensive, $k_2$: cheap, $k_3$: earthquake-resistant, $k_4$: good location, $k_5$: good workmanship. The 6-soft set defined on $R$ is given by:

$(T, K, 6) = \{<< k_1, \{(r_1, 1), (r_2, 4), (r_3, 5)\} >> , << k_2 , \{(r_1, 5), (r_2, 3), (r_3, 2)\} >>, <<$
$k_3, \{(r_1, 4), (r_2, 1), (r_3, 4)\} >>, << k_4, \{(r_1, 3), (r_2, 2), (r_3, 1)\} >>, <<$
$k_5, \{(r_1, 2), (r_2, 5), (r_3, 3)\} >>\}$

***Table 2. Tabular representation of the 6-soft set*** $(\boldsymbol{T, K, 6})$.

| $\boldsymbol{R}$ | $\boldsymbol{r_1}$ | $\boldsymbol{r_2}$ | $\boldsymbol{r_3}$ |
|---|---|---|---|
| $\boldsymbol{k_1}$ | 1 | 4 | 5 |
| $\boldsymbol{k_2}$ | 5 | 3 | 2 |
| $\boldsymbol{k_3}$ | 4 | 1 | 4 |
| $\boldsymbol{k_4}$ | 3 | 2 | 1 |
| $\boldsymbol{k_5}$ | 2 | 5 | 3 |

**Definition 2.8.** [22] Let $(T_1, K, N)$ and $(T_2, K, N)$ be two N-soft sets defined over a universal set $R$, with a common parameter set $K$, and let $C = \{0,1,2,3, \dots, N-1\}$ denote the set of possible scores. For each parameter $k \in K$ and for all $r \in R$, let the score assigned to $r$ under $T_1(k)$ be denoted by $l_{T_1}(k, r)$, and similarly, let the score under $T_2(k)$ be denoted by $l_{T_2}(k, r)$. If the following condition holds: $l_{T_1}(k, r) \leq l_{T_2}(k, r)\ for\ all\ k \in K, r \in R$, then $(T_1, K, N)$ is said to be an N-soft subset of $(T_2, K, N)$, denoted as: $(T_1, K, N) \cong (T_2, K, N)$. In this context, the comparison is made elementwise between the scores assigned to each object $r$ by both soft sets under the same parameter $k$. That is, if $\left(r, l_{T_1}(k, r)\right) \in T_1(k)$ and $\left(r, l_{T_2}(k, r)\right) \in T_2(k)$, then the inequality $l_{T_1}(k, r) \leq l_{T_2}(k, r)$ must be satisfied.

**Example 2.9.** Let the universe set be $R = \{r_1, r_2, r_3, r_4\}$ and the set of parameters be $K = \{k_1, k_2, k_3, k_4, k_5\}$. The 6-soft sets $(T_1, K, 6)$ and $(T_2, K, 6)$ are given in Table 3 and Table 4, respectively. In this case, we have $(T_1, K, 6) \cong (T_2, K, 6)$.

***Table 3. 6-soft set*** $(T_1, K, 6)$

| $R$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ |
|---|---|---|---|---|
| $k_1$ | 4 | 3 | 3 | 1 |
| $k_2$ | 1 | 3 | 4 | 1 |
| $k_3$ | 4 | 2 | 1 | 3 |
| $k_4$ | 2 | 5 | 1 | 1 |
| $k_5$ | 2 | 4 | 3 | 4 |

***Table 4. 6-soft set*** $(T_2, K, 6)$

| $R$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ |
|---|---|---|---|---|
| $k_1$ | 5 | 5 | 4 | 4 |
| $k_2$ | 4 | 5 | 5 | 5 |
| $k_3$ | 5 | 4 | 3 | 4 |
| $k_4$ | 4 | 5 | 5 | 2 |
| $k_5$ | 3 | 5 | 4 | 4 |

**Definition 2.12.** [22] Let $R$ be the universal set, $K$ the set of parameters, and $C = \{0,1,2,\dots,N-1\}$. Define $T_0: K \to 2^{R\times\{0\}}$ by $T_0(k) = \{(r,0) \mid r \in R\}$ for all $k \in K$. Then the N-soft set $(T_0, K, N)$ defined on $R$ is called the null N-soft set.

**Definition 2.13.** [22] Let $R$ be the universal set, $K$ the set of parameters, and $C = \{0,1,2,\dots,N-1\}$. Define $T_{N-1}: K \to 2^{R\times\{N-1\}}$ by $T_{N-1}(k) = \{(r,N-1) \mid r \in R\}$ $for\ all\ k \in K$. Then the N-soft set $(T_{N-1}, K, N)$ defined on $R$ is called the universal N-soft set.

## 3 N-SOFT TOPOLOGICAL STRUCTURES

Let $R$ be a universal set of objects, $K$ a set of parameters (attributes), and $C = \{0,1,2,\dots,N-1\}$ the set of possible ratings. An N-soft set over $R$ is a mapping $T: K \to 2^{R\times C}$, such that for each $k \in K, T(k) = \{(r, l_k(r)) \mid r \in R, l_k(r) \in C\}$, where each $r \in R$ is assigned exactly one rating under each parameter $k$.

**Definition 3.1.** [22] Let $(T,K,N)$ be an N-soft set. The N-soft power set denoted $\mathcal{P}(T,K,N)$, is the collection of all N-soft subsets of $(T,K,N)$. That is, $\mathcal{P}(T,K,N) = \{(T_i,K,N) \mid (T_i,K,N) \subseteq (T,K,N), i \in I \subseteq \mathbb{N}\}$, where for each $k \in K$ and $r \in R$, the score in the subset satisfies $l_k^i(r) \le l_k(r)$, meaning each element in the subset has scores less than or equal to the original set. The cardinality of the N-soft power set is given by: $|\mathcal{P}(T,K,N)| = 2^{\sum_{k\in K}|T(k)|}$, where $|T(k)|$ is the number of elements (object-score pairs) in the set $T(k)$.

**Definition 3.2.** [22] Let $\mathcal{P}(T,K,N)$ be the N-soft power set. A subcollection $\tau_N \subseteq \mathcal{P}(T,K,N)$ is said to form an N-soft topology on $R$ if it satisfies:

1. $(T_0,K,N) \in \tau_N$ and $(T_{N-1},K,N) \in \tau_N$,
2. The restricted intersection of any two sets in $\tau_N$ is in $\tau_N$,
3. The restricted union of any collection of sets in $\tau_N$ is in $\tau_N$.

Then, the triplet $(R,\tau_N,N)$ is called an N-soft topological space.

**Definition 3.3.** [22] Elements of $\tau_N$ are called N-soft open sets. The complement of an N-soft open set (defined appropriately via dual ratings) is called an N-soft closed set.

**Definition 3.4.** [22] Let $(F,A,N)$ be an N-soft set. The N-soft closure of $F$, denoted $\bar{F}$, is the smallest N-soft closed set containing $F$. The N-soft interior, denoted $F°$, is the largest N-soft open set contained in $F$.

**Definition 3.5.** [22] Let $(T,K,N)$ be a fixed N-soft set, and let $((T,K,N),\tilde{\tau}_1)$ and $((T,K,N),\tilde{\tau}_2)$ be two N-soft topological spaces defined over the same universe $R$ and parameter set $K$.

1. If $\tilde{\tau}_1 \tilde{\subseteq} \tilde{\tau}_2$ or $\tilde{\tau}_2 \tilde{\subseteq} \tilde{\tau}_1$, then the two N-soft topologies are said to be comparable.
2. If $\tilde{\tau}_1 \tilde{\subseteq} \tilde{\tau}_2$, then $\tilde{\tau}_1$ is called a weaker (or coarser) topology than $\tilde{\tau}_2$, and $\tilde{\tau}_2$ is called a stronger (or finer) topology than $\tilde{\tau}_1$.

**Definition 3.6.** [22] Let $(T,K,N)$ be an N-soft set defined over a universal set $R$. The collection $\tilde{\tau}_1 = \{(T_0,K,N),(T,K,N)\}$ is called the N-soft trivial topology (or N-soft indiscrete topology) on $(T,K,N)$. On the other hand, the collection $\tilde{\tau}_2 = \mathcal{P}(T,K,N)$ where $\mathcal{P}(T,K,N)$ denotes the N-soft power set of $(T,K,N)$, is called the N-soft discrete topology on $(T,K,N)$.

**Definition 3.7.** [22] Let $\left((T,K,N),\tilde{\tau}\right)$ be an N-soft topological space and let $\widetilde{\mathrm{B}} \tilde{\subseteq} \tilde{\tau}$ be a subcollection of $\tilde{\tau}$. If every member of $\tilde{\tau}$ can be expressed as a union (in the N-soft sense) of elements from $\widetilde{\mathrm{B}}$, then $\widetilde{\mathrm{B}}$ is called an N-soft basis for the topology $\tilde{\tau}$. Each element of an N-soft basis is itself an N-soft open set.

**Definition 3.8.** [22] Let $((T_1,K,N),\tilde{\tau})$ be an N-soft topological space, and let $(T_2,K,N) \tilde{\subseteq} (T_1,K,N)$. The N-soft interior of the set $(T_2,K,N)$, denoted by $Int((T_2,K,N))$ or alternatively by $(T_2,K,N)°$, is defined as the N-soft union of all N-soft open subsets contained in $(T_2,K,N)$. That is, the interior consists of the largest N-soft open set entirely contained in

$(T_2,K,N)$. Formally, $(T_2,K,N)^\circ = \widetilde{\bigcup}\{(O,K,N)\in\tilde{\tau} \mid (O,K,N) \mathrel{\tilde{\subseteq}} (T_2,K,N)\}$. Each such $(O,K,N)$ is an N-soft open set, and the interior operator $\circ$ is idempotent, monotone, and satisfies $(T_2,K,N)^\circ \mathrel{\tilde{\subseteq}} (T_2,K,N)$.

**Definition 3.9.** [22] Let $((T_1,K,N),\tilde{\tau})$ be an N-soft topological space, and let $(T_2,K,N) \mathrel{\tilde{\subseteq}} (T_1,K,N)$. The N-soft closure of the set $(T_2,K,N)$, denoted by $Cl((T_2,K,N))$ or $\overline{(T_2,K,N)}$, is defined as the N-soft intersection of all N-soft closed supersets of $(T_2,K,N)$. That is, the closure is the smallest N-soft closed set containing $(T_2,K,N)$. Formally, $\overline{(T_2,K,N)} = \widetilde{\bigcap}\{(C,K,N)\in\tilde{\tau}^K \mid (T_2,K,N) \mathrel{\tilde{\subseteq}} (C,K,N)\}$, where $\tilde{\tau}^K$ the family of all N-soft closed sets associated with the topology $\tilde{\tau}$. By construction, $\overline{(T_2,K,N)}$ is itself an N-soft closed set and satisfies: $(T_2,K,N) \mathrel{\tilde{\subseteq}} \overline{(T_2,K,N)}$.

## 4 CONTINUITY AND HOMEOMORPHISM IN N-SOFT SPACES

**Definition 4.1.** Let $((T,K,N),\tilde{\tau})$ be an N-soft topological space. An N-soft point in this space is defined as a pair $\langle\langle k,\{(r,N-1)\}\rangle\rangle$ where $k\in K$, $r\in R$, and $N\in\mathbb{N}$. This construct signifies a localized evaluation in which the parameter $k$ is associated with a singleton soft rating value $(r,N-1)$. The N-soft point encapsulates the minimal information required to characterize a specific decision criterion within the N-soft topological framework. Given an N-soft point $\langle\langle k,\{(r,N-1)\}\rangle\rangle \in (T,K,N)$, an N-soft neighborhood of $\langle\langle k,\{(r,N-1)\}\rangle\rangle$ is defined as an N-soft open set $(T',K,N)\in\tilde{\tau}$ such that $\left\langle\langle k,\{(r,N-1)\}\rangle\right\rangle \in (T',K,N)$. In other words, any N-soft open set containing the point $\langle\langle k,\{(r,N-1)\}\rangle\rangle$ is called an N-soft neighborhood of that point.

**Definition 4.2.** Let $((T_1,K,N),\tilde{\tau}_1)$ and $((T_2,K,N),\tilde{\tau}_2)$ be two N-soft topological spaces, and let $f:((T_1,K,N),\tilde{\tau}_1)\longrightarrow((T_2,K,N),\tilde{\tau}_2)$ be a function. For a given element $\ll k,\{(r,N-1)\}\gg \in (T_1,K,N)$, if for every N-soft neighborhood $\mathcal{N}^*(f(\ll k,\{(r,N-1)\}\gg))$, there exists an N-soft neighborhood $\mathcal{N}^*(\ll k,\{(r,N-1)\}\gg)$ such that $f(\mathcal{N}^*(\ll k,\{(r,N-1)\}\gg)) \mathrel{\tilde{\subseteq}} \mathcal{N}^*(f\ll k,\{(r,N-1)\}\gg)$, then $f$ is called an N-soft continuous function at the point $\ll k,\{(r,N-1)\}\gg$.

**Example 4.3.** Consider the following 5-soft sets: $(T_1,K,5)=\{\ll k_1,\{(r_1,4),(r_2,1)\}\gg,\ll k_2,\{(r_1,3),(r_2,2)\}\gg\}$, and $(T_2,K,5)=\{\ll k_1,\{(r_1,3),(r_2,4)\}\gg,\ll k_2,\{(r_1,2),(r_2,3)\}\gg\}$, with their respective trivial 5-soft topologies: $\tilde{\tau}_1=\{(T_0,K,5),(T_1,K,5)\}$, $\tilde{\tau}_2=\{(T_0,K,5),(T_2,K,5)\}$. Define a function $f:((T_1,K,N),\tilde{\tau}_1)\longrightarrow$

$((T_2,K,N),\tilde{\tau}_2)$ by mapping ratings from the first set to the second set as follows $f\left(\ell^1_{k_1}(r_1)\right) = \left(\ell^2_{k_1}(r_1)\right) = 3$, $f\left(\ell^1_{k_1}(r_2)\right) = \left(\ell^2_{k_1}(r_2)\right) = 4$, $f\left(\ell^1_{k_2}(r_1)\right) = \left(\ell^2_{k_2}(r_1)\right) = 2$, $f\left(\ell^1_{k_2}(r_2)\right) = \left(\ell^2_{k_2}(r_2)\right) = 2$ where $\ell^1_k(r)$, $\ell^2_k(r)$ represent the ratings assigned by the 5-soft sets $T_1(k)$ and $T_2(k)$, respectively. To verify the continuity of $f$ at the point $l^1_{k_2}(r_1) \in (T_1,K,5)$, note that $f(l^1_{k_2}(r_1)) = 2$. Thus, the corresponding 5-soft neighborhood is: $N^*\left(f\left(l^1_{k_2}(r_1)\right)\right) = \{(T_2,K,5)\}$. Since $(T_2,K,5) \in N^*\left(f\left(l^1_{k_2}(r_1)\right)\right)$, choose $(T_3,K,N) = \{\ell^1_{k_2}(r_1)\} \in \mathcal{N}^*\left(\ell^1_{k_2}(r_1)\right)$. Clearly, we have: $f\left((T_3,K,5)\right) \tilde{\subseteq} \{(T_2,K,5)\}$, indicating that $f$ is 5-soft continuous at the point $l^1_{k_2}(r_1)$.

**Proposition 4.4.** Let $f\colon \left((T_1,K,N),\tilde{\tau}_1\right) \to ((T_2,K,N),\tilde{\tau}_2)$ be a function. The following statements are equivalent for a point $l^1_k(r) \in (T_1,K,N)$:

1. The function $f$ is N-soft continuous at the point $l^1_k(r)$.
2. For every N-soft neighborhood $N \in N^*(f(l^1_k(r)))$, there exists an N-soft neighborhood $(T_3,K,N) \in N^*\,(l^1_k(r))$ such that: $(T_3,K,N) \tilde{\subseteq} f^{-1}\,(N)$.
3. For every N-soft neighborhood $N \in N^*(f(l^1_k(r)))$, the inverse image satisfies: $f^{-1}(N) \in N^*(l^1_k(r))$.

**Proof.** (1) ⇒ (2): Assume $N \in N^*(f(l^1_k(r)))$. By (1), there exists an N-soft neighborhood $(T_3,K,N) \in N^*(l^1_k(r))$ such that: $f((T_3,K,N)) \tilde{\subseteq} N$. Thus, we obtain: $(T_3,K,N) \tilde{\subseteq} f^{-1}(N)$.

(2) ⇒ (3): Suppose $N \in N^*(f(l^1_k(r)))$. By assumption (2), there exists an N-soft neighborhood $(T_3,K,N) \in N^*(l^1_k(r))$ satisfying: $(T_3,K,N) \tilde{\subseteq} f^{-1}(N)$. Using the third axiom of neighborhoods, we immediately conclude: $f^{-1}(N) \in N^*(l^1_k(r))$.

(3) ⇒ (1): Let $N \in N^*(f(l^1_k(r)))$. To show (1), we must find $(T_3,K,N) \in N^*(l^1_k(r))$ such that: $f((T_3,K,N)) \tilde{\subseteq} N$. By hypothesis (3), we have: $f^{-1}(N) \in N^*(l^1_k(r))$. Thus, there exists an N-soft open set $((T_4,K,N) \in \tilde{\tau}_1$ with: $l^1_k(r) \in (T_4,K,N) \tilde{\subseteq} f^{-1}(N)$. If we choose $(T_3,K,N) = (T_4,K,N)$, it follows directly that: $(T_3,K,N) \in N^*\left(l^1_k(r)\right)$ and $f\left((T_3,K,N)\right) \tilde{\subseteq} N$. Hence, (1) holds, completing the proof.

**Remark 4.5.** Up to this point, the discussion has focused on the pointwise definition of N-soft continuity, that is, the continuity of a function at individual points of an N-soft

topological space. In what follows, we aim to establish characterizations of global **N**-soft continuity, which refers to continuity defined on the entire N-soft topological structure. These characterizations provide the necessary and sufficient conditions for a function to be globally N-soft continuous in terms of the preimage of open and closed sets.

**Theorem 4.6.** Let $f:\big((T_1,K,N),\tilde{\tau}_1\big)\to\big((T_2,K,N),\tilde{\tau}_2\big)$ be a function between two N-soft topological spaces. Then the following statements are equivalent:

The function $f$ is N-soft continuous at every point $l_k^1\,(r)\in(T_1,K,N)$,

1. For every N-soft open set $((T_3,K,N)\in\tilde{\tau}_2$, the preimage $f^{-1}\big((T_3,K,N)\big)$ belongs to $\tilde{\tau}_1$.
2. For every N-soft closed set $(T_4,K,N)$, such that $(T_2,K,N)\setminus(T_4,K,N)\in\tilde{\tau}_2$, it follows that $f^{-1}\big((T_4,K,N)\big)\in\tilde{\tau}_1^K$, where $\tilde{\tau}^K$ denotes the collection $\{(F,K,N)\subseteq(T,K,N)\mid (T,K,N)\setminus(F,K,N)\in\tilde{\tau}\}$.

**Proof.** $(1)\Rightarrow(2)$: Assume that $f$ is N-soft continuous at every point $l_k^1(r)\in(T_1,K,N)$. Let $(T_3,K,N)\in\tilde{\tau}_2$ be arbitrary. To prove that $f^{-1}\big((T_3,K,N)\big)\in\tilde{\tau}_1$, we need to show that every point in the preimage is an interior point. Take any $l_k^1(r)\in f^{-1}((T_3,K,N))$. Then $f\big(l_k^1(r)\big)\in(T_3,K,N)$, hence: $(T_3,K,N)\in N^*(f(l_k^1(r)))$. By the hypothesis of pointwise continuity: $f^{-1}\big((T_3,K,N)\big)\in N^*(l_k^1(r))$. Therefore, $f^{-1}((T_3,K,N))$ is open in $\tilde{\tau}_1$.

$(2)\Rightarrow(3)$: Let $(T_4,K,N)$ be such that $(T_2,K,N)\setminus(T_4,K,N)\in\tilde{\tau}_2$. By (2), it follows that: $f^{-1}\big((T_2,K,N)\setminus(T_4,K,N)\big)\in\tilde{\tau}_1$. Since $f^{-1}\big((T_2,K,N)\big)\setminus f^{-1}\big((T_4,K,N)\big)=(T_1,K,N)\setminus f^{-1}((T_4,K,N))$, we conclude that $f^{-1}\big((T_4,K,N)\big)\in\tilde{\tau}_1^K$.

$(3)\Rightarrow(1)$: Let $l_k^1(r)\in(T_1,K,N)$, and suppose that $N\in N^*(f(l_k^1(r)))$. Then there exists an open set $(T_5,K,N)\in\tilde{\tau}_2$ such that: $f\big(l_k^1(r)\big)\in(T_5,K,N)\,\tilde{\subseteq}\,N$. Consequently, $(T_2,K,N)\setminus(T_5,K,N)\in\tilde{\tau}_2^K$. By assumption (3), it follows that: $f^{-1}\big((T_2,K,N)\setminus(T_5,K,N)\big)\in\tilde{\tau}_1^K$. Therefore,

$$(T_1,K,N)\setminus f^{-1}\big((T_2,K,N)\setminus(T_5,K,N)\big)=f^{-1}\big((T_5,K,N)\big)\in\tilde{\tau}_1.$$

Since $f\big(l_k^1(r)\big)\in(T_5,K,N)\,\tilde{\subseteq}\,N$, we obtain: $l_k^1(r)\in f^{-1}\big((T_5,K,N)\big)\,\tilde{\subseteq}\,f^{-1}(N)$, which implies that $f^{-1}(N)\in N^*(l_k^1(r))$. Thus, $f$ is N-soft continuous at the point $l_k^1(r)$. This concludes the proof.

**Theorem 4.7.** Let $f:\left((T_1,K,N),\tilde{\tau}_1\right)\to((T_2,K,N),\tilde{\tau}_2)$ be a function. Then the following statements are equivalent:

1. $f$ is an N-soft continuous function.
2. For every $(T_3,K,N)\ \tilde{\subseteq}\ (T_1,K,N)$, we have $f\left(\overline{(T_3,K,N)}\right)\ \tilde{\subseteq}\ \overline{\left(f((T_3,K,N))\right)}$.
3. For every $(T_4,K,N)\ \tilde{\subseteq}\ (T_2,K,N)$, the following holds:

$$\overline{\left(f^{-1}\left((T_4,K,N)\right)\right)}\ \tilde{\subseteq}\ f^{-1}\left(\overline{(T_4,K,N)}\right).$$

4. For every $(T_5,K,N)\ \tilde{\subseteq}\ (T_2,K,N)$, the following holds:

$$f^{-1}((T_5,K,N)^{\circ})\ \tilde{\subseteq}\ \left(f^{-1}(T_5,K,N)\right)^{\circ}.$$

**Proof.** (1) ⇒ (2): Let $(T_3,K,N)\ \tilde{\subseteq}\ (T_1,K,N)$ and suppose $l_k^2(r)\in f\left(\overline{(T_3,K,N)}\right)$. Then there exists $l_k^1(r)\in\overline{(T_3,K,N)}$ such that $f(l_k^1(r))=l_k^2(r)$. Assume, for contradiction, that $f\left(l_k^1(r)\right)\notin f\left(\overline{(T_3,K,N)}\right)$. Then by the definition of the N-soft closure, there exists an open set $(T_6,K,N)\in\tilde{\tau}_2$ such that: $f\left(l_k^1(r)\right)\in(T_6,K,N)\ and\ (T_6,K,N)\tilde{\cap}\ f\left((T_3,K,N)\right)=(T_0,K,N)$. Taking preimages, we get: $f^{-1}((T_6,K,N))\ \tilde{\cap}\ f^{-1}\left(f((T_3,K,N))\right)=(T_0,K,N)$. Since $f^{-1}(f((T_3,K,N)))\ \tilde{\supseteq}\ (T_3,K,N)$, it follows that: $f^{-1}\left((T_6,K,N)\right)\tilde{\cap}\ (T_3,K,N)=(T_0,K,N)$. Because $f$ is N-soft continuous, the preimage $f^{-1}\left((T_6,K,N)\right)\in\tilde{\tau}_1$. Hence, $l_k^1(r)$ is not in $\overline{(T_3,K,N)}$, which contradicts the assumption. Therefore, $f\left(\overline{(T_3,K,N)}\right)\ \tilde{\subseteq}\ \overline{\left(f((T_3,K,N))\right)}$.

(2) ⇒ (3): Let $(T_4,K,N)\ \tilde{\subseteq}\ (T_2,K,N)$. Then $f^{-1}((T_4,K,N))\ \tilde{\subseteq}\ (T_1,K,N)$. By the hypothesis in (2), we have: $f\left(\overline{f^{-1}\left((T_4,K,N)\right)}\right)\ \tilde{\subseteq}\ \overline{f\left(f^{-1}\left((T_4,K,N)\right)\right)}\ \tilde{\subseteq}\ \overline{(T_4,K,N)}$. Taking the preimage of both sides gives:

$$f^{-1}\left(f\left(\overline{f^{-1}\left((T_4,K,N)\right)}\right)\right)\ \tilde{\subseteq}\ f^{-1}\left(\overline{(T_4,K,N)}\right).$$

Since $\overline{f^{-1}\left((T_4,K,N)\right)}\ \tilde{\subseteq}\ f^{-1}(f(\overline{f^{-1}\left((T_4,K,N)\right)}))$, it follows that:

$$\overline{f^{-1}\left((T_4,K,N)\right)}\ \tilde{\subseteq}\ f^{-1}(\overline{(T_4,K,N)}).$$

(3) ⇒ (4): Let $(T_5,K,N)\ \tilde{\subseteq}\ (T_2,K,N)$. Then $(T_2,K,N)\setminus(T_5,K,N)\ \tilde{\subseteq}\ (T_2,K,N)$. From (3), we have:

$$\overline{f^{-1}\big((T_2,K,N)\backslash(T_5,K,N)\big)} \cong f^{-1}\big(\overline{(T_2,K,N)\backslash(T_5,K,N)}\big)$$

But $\overline{(T_2,K,N)\backslash(T_5,K,N)} = (T_2,K,N) \setminus (T_5,K,N)^{\circ}$, so:

$$f^{-1}\big((T_2,K,N) \setminus (T_5,K,N)^{\circ}\big) = (T_1,K,N) \setminus f^{-1}((T_5,K,N)^{\circ}).$$

Hence, the complement of the left-hand side yields:

$$(T_1,K,N) \setminus \overline{f^{-1}\big((T_2,K,N) \setminus (T_5,K,N)\big)} \supseteq \Big(f^{-1}\big((T_5,K,N)\big)\Big)^{\circ}.$$

Thus, $f^{-1}((T_5,K,N)^{\circ}) \cong \big(f^{-1}(T_5,K,N)\big)^{\circ}$.

$(4) \Rightarrow (1)$: Let $(T_6,K,N) \in \tilde{\tau}_2$. Then $(T_6,K,N) = (T_6,K,N)^{\circ}$ since it is N-soft open. By hypothesis,

$$f^{-1}((T_6,K,N)) = f^{-1}((T_6,K,N)^{\circ}) \cong \Big(f^{-1}\big((T_6,K,N)\big)\Big)^{\circ}.$$

This implies that $f^{-1}((T_6,K,N))$ contains an N-soft open set and is therefore open in $\tilde{\tau}_1$. Hence, $f$ is N-soft continuous. This completes the proof.

**Definition 4.8.** Let $f:\big((T_1,K,N),\tilde{\tau}_1\big) \to ((T_2,K,N),\tilde{\tau}_2)$ be a function. If the images of any two distinct elements of $(T_1,K,N)$ are also distinct, then $f$ is said to be N-soft injective (or N-soft one-to-one). Symbolically, this can be expressed as: $f$ is N-soft injective $\Leftrightarrow \forall l_k^*(r), l_k^{**}(r) \in (T_1,K,N), l_k^*(r) \neq l_k^{**}(r) \Rightarrow f\big(l_k^*(r)\big) \neq f(l_k^{**}(r))$. Equivalently, $f$ is N-soft injective $\Leftrightarrow \forall l_k^*(r), l_k^{**}(r) \in (T_1,K,N),\ f\big(l_k^*(r)\big) = f\big(l_k^{**}(r)\big) \Rightarrow l_k^*(r) = l_k^{**}(r)$.

**Definition 4.9.** Let $f:\big((T_1,K,N),\tilde{\tau}_1\big) \to ((T_2,K,N),\tilde{\tau}_2)$ be a function. If for every element $l_k^2(r) \in (T_2,K,N)$, there exists at least one element $l_k^1(r) \in (T_1,K,N)$ such that $f(l_k^1(r)) = l_k^2(r)$, then $f$ is called an N-soft surjective (or N-soft onto) function. Symbolically: $f$ is N-soft surjective $\Leftrightarrow \forall l_k^2(r) \in (T_2,K,N),\ \exists l_k^1(r) \in (T_1,K,N)$ such that $f\big(l_k^1(r)\big) = l_k^2(r)$. If $f$ is both N-soft injective and N-soft surjective, then $f$ is called an N-soft bijective function (or N-soft one-to-one correspondence).

**Definition 4.10.** Let $((T_1,K,N),\tilde{\tau}_1)$ and $((T_2,K,N),\tilde{\tau}_2)$ be N-soft topological spaces. A bijective N-soft function $f:((T_1,K,N),\tilde{\tau}_1) \longrightarrow ((T_2,K,N),\tilde{\tau}_2)$ is said to be an N-soft homeomorphism if both $f$ and its inverse $f^{-1}$ are N-soft continuous. In this case, the spaces $\big((T_1,K,N),\tilde{\tau}_1\big)$ and $((T_2,K,N),\tilde{\tau}_2)$ are called N-soft homeomorphic, or N-soft topologically equivalent, which is denoted by $\big((T_1,K,N),\tilde{\tau}_1\big) \approx ((T_2,K,N),\tilde{\tau}_2)$. This notation signifies that the two N-soft topological spaces possess identical topological structure up to homeomorphism.

**Remark 4.11.** If $f:((T_1,K,N),\tilde{\tau}_1)\longrightarrow((T_2,K,N),\tilde{\tau}_2)$ is bijective and its inverse $f^{-1}$ is N-soft continuous, then for every N-soft open set $(T,K,N)\in\tilde{\tau}_1$, the image $f\big((T,K,N)\big)=(f^{-1})^{-1}\big((T,K,N)\big)\in\tilde{\tau}_2$. This implies that $f$ is an N-soft open mapping. Similarly, if $(T,K,N)\in\tilde{\tau}_1^K$ is an N-soft closed set, then $f\big((T,K,N)\big)=(f^{-1})^{-1}\big((T,K,N)\big)\in\tilde{\tau}_2^K$, which implies that $f$ is also an N-soft closed mapping. Therefore, if $f$ is a bijective N-soft continuous function and its inverse $f^{-1}$ is also N-soft continuous, then $f$ is an N-soft homeomorphism.

**Theorem 4.12.** Let $f:((T_1,K,N),\tilde{\tau}_1)\longrightarrow((T_2,K,N),\tilde{\tau}_2)$ be an N-soft bijective function. Then the following statements are equivalent:

1. $f$ is an N-soft homeomorphism.
2. $f$ is N-soft continuous and an open mapping.
3. $f$ is N-soft continuous and a closed mapping.
4. For every $(T_3,K,N)\tilde{\subseteq}(T_1,K,N)$, we have: $f(\overline{(T_3,K,N)})=\overline{f((T_3,K,N))}$.
5. For every $(T_4,K,N)\tilde{\subseteq}(T_2,K,N)$, we have: $\overline{f^{-1}((T_4,K,N))}=f^{-1}\big(\overline{(T_4,K,N)}\big)$.
6. For every $(T_4,K,N)\tilde{\subseteq}(T_2,K,N)$, we have: $\Big(f^{-1}\big((T_4,K,N)\big)\Big)^{\circ}=f^{-1}((T_4,K,N)^{\circ})$.

**Proof.** (1) ⇒ (2): Assume that $f$ is an N-soft homeomorphism. Then $f$ is N-soft continuous by definition. From Remark 4.11, any homeomorphism is also an open mapping, hence $f$ is N-soft open.

(2) ⇒ (3): Suppose $f$ is N-soft continuous and N-soft open. Let $(T,K,N)\in\tilde{\tau}_1^K$ be any N-soft closed set. Then its complement $(T_1,K,N)\setminus(T,K,N)\in\tilde{\tau}_1$. Since $f$ is open and bijective, we have: $f\big((T_1,K,N)\setminus(T,K,N)\big)=(T_2,K,N)\setminus f\big((T,K,N)\big)\in\tilde{\tau}_2$. This implies $f\big((T,K,N)\big)\in\tilde{\tau}_2^K$, showing that $f$ is N-soft closed.

(3) ⇒ (4): Let $f$ be N-soft continuous and closed. For any $(T_3,K,N)\tilde{\subseteq}(T_1,K,N)$, Theorem 4.7. yields: $f\big(\overline{(T_3,K,N)}\big)\tilde{\subseteq}\overline{\big(f(T_3,K,N)\big)}$ ... $(I)$. Meanwhile, since $f$ is N-soft closed, it preserves closure in the reverse inclusion: $f\big(\overline{(T_3,K,N)}\big)\tilde{\supseteq}\overline{\big(f((T_3,K,N))\big)}$ ... $(II)$. From $(I)$ and $(II)$, we conclude: $f\big(\overline{(T_3,K,N)}\big)=\overline{\big(f(T_3,K,N)\big)}$.

(4) ⇒ (5): Let $(T_4,K,N)\tilde{\subseteq}(T_2,K,N)$. Then $f^{-1}((T_4,K,N))\tilde{\subseteq}(T_1,K,N)$. Applying the hypothesis, we get: $f\left(\overline{f^{-1}\big((T_4,K,N)\big)}\right)=\overline{f\left(f^{-1}\big((T_4,K,N)\big)\right)}=\overline{(T_4,K,N)}$. Therefore: $\overline{f^{-1}\big((T_4,K,N)\big)}=f^{-1}(\overline{(T_4,K,N)})$.

(5) ⇒ (6): Let $(T_4,K,N) \cong (T_2,K,N)$. Then $(T_2,K,N)\backslash(T_4,K,N) \cong (T_2,K,N)$. Applying the hypothesis: $\overline{f^{-1}((T_2,K,N) \setminus (T_4,K,N))} = f^{-1}\big(\overline{(T_2,K,N) \setminus (T_4,K,N)}\big)$. Note that: $\overline{(T_2,K,N) \setminus (T_4,K,N)} = (T_2,K,N) \setminus (T_4,K,N)^\circ$. Hence:

$f^{-1}\big((T_2,K,N) \setminus (T_4,K,N)\big) = (T_1,K,N) \setminus f^{-1}((T_4,K,N))$ and

$\overline{(T_1,K,N) \setminus f^{-1}\big((T_4,K,N)\big)} = (T_1,K,N) \setminus \Big(f^{-1}\big((T_4,K,N)\big)\Big)^\circ$. Combining both, we derive:

$$(T_1,K,N) \setminus \Big(f^{-1}\big((T_4,K,N)\big)\Big)^\circ = (T_1,K,N) \setminus f^{-1}((T_4,K,N)^\circ)$$

which implies: $\Big(f^{-1}\big((T_4,K,N)\big)\Big)^\circ = f^{-1}((T_4,K,N)^\circ)$.

(6) ⇒ (1): To prove that $f$ is an N-soft homeomorphism, it suffices to show that both $f$ and $f^{-1}$ are N-soft continuous. Let $(T_5,K,N) \in \tilde{\tau}_2$. From the hypothesis:

$$\Big(f^{-1}\big((T_5,K,N)\big)\Big)^\circ = f^{-1}((T_5,K,N)^\circ) = f^{-1}\big((T_5,K,N)\big).$$

Thus, $f^{-1}((T_5,K,N))$ is N-soft open, i.e., it belongs to $\tilde{\tau}_1$. By Theorem 4.6., $f$ is N-soft continuous. Now, consider $(T,K,N) \in \tilde{\tau}_1$. To show that $f^{-1}$ is continuous, we must check whether: $f\big((T,K,N)\big) \in \tilde{\tau}_2$. From the hypothesis:

$$\Big(f^{-1}\Big(f\big((T,K,N)\big)\Big)\Big)^\circ = f^{-1}\big(f\big((T,K,N)\big)^\circ\big)$$

Since $(T,K,N)^\circ = (T,K,N)$, it follows that:

$$(T,K,N) = f^{-1}\big(f\big((T,K,N)\big)^\circ\big) \Rightarrow f((T,K,N)) = \Big(f\big((T,K,N)\big)\Big)^\circ$$

Thus, $f\big((T,K,N)\big) \in \tilde{\tau}_2$, and $f^{-1}$ is N-soft continuous. This completes the proof.

**Example 4.13.** Let $(T,K,5)$ be a 5-soft set defined as:

$$(T,K,5) = \{\langle\langle k_1,\{(r_1,4),(r_2,1)\}\rangle\rangle,\langle\langle k_2,\{(r_1,3)\}\rangle\rangle\}.$$

Define the 5-soft topologies $\tilde{\tau}_1$ and $\tilde{\tau}_2$ on $(T,K,5)$ as:

$$\tilde{\tau}_1 = \{T_0,T_1,T_2,T_3,T_4,T_5\},\ \ \tilde{\tau}_2 = \{T_0^*,T_1^*,T_2^*,T_3^*,T_4^*,T_5^*\},$$

where:

- $T_0 = T_0^* = \emptyset$,
- $T_5 = T_5^* = (T,K,5)$,
- $T_1 = \{\langle\langle k_1,\{(r_1,4)\}\rangle\rangle\}$,

- $T_2 = \{\langle\langle\, k_1, \{(r_2, 1)\}\,\rangle\rangle\}$,
- $T_3 = T_1 \,\tilde{\cup}\, T_2 = \{\langle\langle\, k_1, \{(r_1, 4), (r_2, 1)\}\,\rangle\rangle\}$,
- $T_4 = \{\langle\langle\, k_1, \{(r_2, 1)\}\,\rangle\rangle, \langle\langle\, k_2, \{(r_1, 3)\}\rangle\rangle\}$,
- $T_1^* = \{\langle\langle\, k_1, \{(r_2, 1)\}\,\rangle\rangle\}$,
- $T_2^* = \{\langle\langle\, k_2, \{(r_1, 3)\}\,\rangle\rangle\}$,
- $T_3^* = \{\langle\langle\, k_1, \{(r_1, 4), (r_2, 1)\}\,\rangle\rangle\}$,
- $T_4^* = T_1^* \,\tilde{\cup}\, T_2^* = \{\langle\langle\, k_1, \{(r_2, 1)\}\,\rangle\rangle, \langle\langle\, k_2, \{(r_1, 3)\}\,\rangle\rangle\}$.

Define a function $f : \big((T, K, 5), \tilde{\tau}_1\big) \to ((T, K, 5), \tilde{\tau}_2)$ by mapping individual score values as:

$$f(4) = 3, \qquad f(1) = 1, \qquad f(3) = 4.$$

That is, $f$ acts on the ratings of each pair $(r_i, s)$ by transforming the score $s$ according to the above rule. Clearly, $f$ is bijective on the 5-soft set $(T, K, 5)$, since each distinct value is mapped to a distinct one and every value in the codomain is covered. Furthermore, since for every open set $(T_i^*, K, 5) \in \tilde{\tau}_2$, its preimage $f^{-1}\big((T_i^*, K, 5)\big) \in \tilde{\tau}_1$, the function $f$ is N-soft continuous by Theorem 3.26. Similarly, because $f$ maps N-soft open sets in $\tilde{\tau}_1$ to N-soft open sets in $\tilde{\tau}_2$, $f$ is also an N-soft open mapping. Therefore, by Theorem 4.11., $f$ is an N-soft homeomorphism between $((T, K, 5), \tilde{\tau}_1)$ and $((T, K, 5), \tilde{\tau}_2)$.

## 5 APPLICATION OF N-SOFT HOMEOMORPHISM IN HEALTH INDICATOR-BASED DECISION SYSTEMS

This section introduces a practical application of N-soft sets and N-soft topology by modeling real-world public health data through a formalized decision-theoretic framework. The dataset is drawn from the Behavioral Risk Factor Surveillance System (BRFSS) [26], a comprehensive health survey conducted by the Centers for Disease Control and Prevention. We restrict our analysis to adults aged 35-44, a demographic particularly relevant for preventive health strategies. The universe of discourse consists of a subset of U.S. states: $U = \{\text{Alabama}, \text{Alaska}, \text{Arizona}, \text{Arkansas}, \text{California}\}$. The parameter set $E$ consists of multi-criteria health indicators selected from BRFSS, specifically $E = \{e_1 = Obesity, e_2 = Physical\ Activity,\ e_3 = Fruit\ Intake, e_4 = Vegetable\ Intake, e_5 = Inactivity\}$. Each $e_j \in E$ corresponds to a measurable health-related behavior or outcome. Indicators with negative public health connotations (e.g., obesity, inactivity, and insufficient consumption of fruits or vegetables) are normalized inversely to ensure that higher scores represent better health

status. All values are scaled into the interval [0,1] using Min-Max normalization. Let $R \subseteq [0,1]$ be the set of these normalized scores. An N-soft set is defined over the triplet $(U, E, R)$ via a mapping: $F: E \to 2^{U \times R}, F(e_j) = \{(u_i, r_{ij}) \mid u_i \in U, r_{ij} \in R\}$. This mapping assigns each object–parameter pair $(u_i, e_j)$ a normalized value $r_{ij}$, forming the basis of a tabular representation. Table 5 presents the normalized scores associated with each state–parameter pair, constituting the complete N-soft representation of the dataset.

***Table 5. N-soft representation based on normalized health indicators***

| | Obesity | Physical Activity | Fruit Intake | Vegetable Intake | Inactivity |
|---|---|---|---|---|---|
| Alabama | 0.07 | 0.75 | 0.05 | 0.50 | 0.29 |
| Alaska | 0.39 | 1.00 | 0.39 | 1.00 | 1.00 |
| Arizona | 0.66 | 0.09 | 0.47 | 0.21 | 0.97 |
| Arkansas | 0.00 | 0.65 | 0.00 | 0.37 | 0.00 |
| California | 1.00 | 0.00 | 1.00 | 0.00 | 0.83 |

To construct an N-soft topology over this structure, we define threshold-based N-soft open sets. For each $e_j \in E$ and a threshold $\theta_j \in [0,1]$, an N-soft open set is defined as: $A_j^{\theta_j} = \{u_i \in U \mid F(e_j)(u_i) > \theta_j\}$. For example, $A_3^{0.7} = \{California\}$, since only California scores above 0.7 on fruit intake; $A_2^{0.6} = \{Alabama, Alaska, Arkansas\}$, based on physical activity. These open sets generate a basis $\mathcal{B}$, and the corresponding N-soft topology $\tau$ is constructed as: $\tau = \{\cup A_j^{\theta_j} \mid A_j^{\theta_j} \subseteq \mathcal{B}\}$. Two distinct N-soft topologies are constructed to represent different policy scenarios: $\tau_{eq}$: all indicators weighted equally; thresholds applied uniformly. $\tau_{prio}$: fruit intake and physical activity prioritized via stricter thresholds. To examine structural consistency under changing priorities, we define a mapping $f: (U, \tau_{eq}) \to (U, \tau_{prio})$ that transforms equal-weighted scores into priority-weighted scores. For each state $p \in U$, we define: $f(p) = \sum_{j=1}^{5} w_j^{prio} \cdot r_{pj}$, $f^{-1}(p) = \sum_{j=1}^{5} w_j^{eq} \cdot r_{pj}$, where: $w_j^{eq} = 0.25$ for all $j$, $w_j^{prio} = \{0.2, 0.6, 0.1, 0.1\}$ for obesity, physical activity, fruit, and vegetable intake, respectively. Table 6 presents the aggregated scores obtained using both weighting schemes.

***Table 6. Aggregated health scores under equal and priority weighting***

| | Fruit | Vegetable | Physical Activity | 1-Obesity | Score_Eq | Score_Priority |
|---|---|---|---|---|---|---|
| Alabama | 1.0 | 0.25 | 0.89 | 0.08 | 0.55 | 0.67 |
| Alaska | 0.81 | 0.0 | 0.16 | 0.49 | 0.37 | 0.28 |
| Arizona | 0.74 | 1.0 | 1.0 | 0.67 | 0.85 | 0.91 |
| Arkansas | 1.0 | 0.32 | 0.34 | 0.0 | 0.42 | 0.34 |
| California | 0.0 | 0.71 | 0.0 | 1.0 | 0.43 | 0.27 |

To verify that $f$ is an N-soft homeomorphism, we establish the following properties. First, bijectivity is satisfied since each state $u_i \in U$ is mapped to a unique scalar score determined by fixed weights over distinct indicator vectors. If $f(u_i) = f(u_k)$, then $\sum_j w_j^{prio} \cdot r_{ij} = \sum_j w_j^{prio} \cdot r_{kj}$, which implies $r_{ij} = r_{kj}$ for all $j$, and hence $u_i = u_k$. Second, continuity is guaranteed by the linearity and boundedness of the score function. Given that the topologies are generated via threshold-based open sets $A_j^{\theta_j} = \{ u_i \in U \mid r_{ij} > \theta_j \}$, the image of an open set under $f$ remains open in $\tau_{prio}$ due to the stability of inequalities under affine transformation. Formally, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\| r_{ij} - r_{kj} \| < \delta \Rightarrow |f(u_i) - f(u_k)| < \epsilon$, which follows from the uniform continuity of linear maps over compact domains in $\mathbb{R}^m$. Third, since the inverse function $f^{-1}$ is also an affine combination over the same domain with fixed weights, the same continuity property holds in reverse, ensuring inverse continuity. Lastly, topological operations such as union and intersection of open sets are preserved under $f$ and $f^{-1}$. For example, consider $A_2^{0.3} = \{ u_i \in U \mid r_{i2} > 0.3 \}$ and $A_3^{0.5} = \{ u_i \in U \mid r_{i3} > 0.5 \}$, then $A_2^{0.3} \cup A_3^{0.5}$ and $A_2^{0.3} \cap A_3^{0.5}$ remain open in both topologies, and their images and preimages under $f$ correspond to sets defined by linear inequalities on the score space. To illustrate how the N-soft homeomorphism preserves the structural relationships between elements under different weighting schemes, consider the case of Arizona. In Table 6, Arizona has a high normalized fruit intake (0.74), vegetable intake (1.0), physical activity (1.0), and low obesity (inverse normalized: 0.67). Under the equal-weighted scheme, it receives a composite score of 0.85; under the priority-weighted model, where physical activity and fruit intake are emphasized, its score increases to 0.91. Despite the difference in scalar values, Arizona remains the top-ranked state under both scoring functions. This outcome reflects that the ordering and relational structure among states is preserved under the mapping $f$, thereby affirming the homeomorphic equivalence of the two topologies. It further confirms that while the weighting scheme alters the emphasis, it does not distort the internal configuration of the decision space. Moreover, the selection of threshold values $\theta_j$ plays a critical role in ensuring valid topology. While the current construction uses moderate thresholds (e.g., 0.5-0.7), extreme or inconsistent thresholding may result in loss of openness preservation. Hence, the equivalence shown here holds under coherent and policy-guided threshold strategies, but should not be presumed for arbitrary settings.

All steps, including normalization, score aggregation, construction of threshold-based open sets, and verification of topological properties, were carried out using Python's *pandas* and *numpy* libraries. This guarantees full traceability and mathematical consistency between the symbolic structure of N-soft topologies and their empirical instantiations. Accordingly, the mapping $f$ satisfies all conditions of an N-soft homeomorphism, providing a mathematically rigorous foundation for the structural equivalence of the two decision models. This topological equivalence indicates that the structure of the decision space remains intact even when the weighting scheme is altered. Nevertheless, it should be noted that such equivalence critically depends on the specific properties of the transformation $f$. The bijectivity and continuity requirements are fulfilled due to the linear construction and the bounded nature of the input domain. However, if $f$ were nonlinear or discontinuous, or if the weighting scheme introduced degeneracy (e.g., producing identical scores for distinct states), then the preservation of open sets or neighborhood structures might fail. Therefore, while the current setting ensures a valid N-soft homeomorphism, this property should not be presumed to hold for arbitrary transformations. In conclusion, the application of N-soft topology offers a robust and flexible mathematical framework for modeling adaptive decision-making systems in public health. Its ability to accommodate shifting policy priorities without disrupting the internal structure of the evaluation space underscores both the theoretical soundness and practical utility of the N-soft homeomorphism concept.

## 6 CONCLUSION

The concept of N-soft homeomorphism guarantees that decision spaces remain topologically equivalent with respect to preferences, threshold levels, and scoring mechanisms. This structural equivalence provides a solid foundation for designing decision algorithms that are robust, generalizable, and reusable across different contexts. Integrating N-soft homeomorphism into decision-making frameworks offers a novel and mathematically rigorous approach to maintaining consistency and adaptability in multi-criteria environments. By preserving the underlying topological properties of evaluation systems, it enables decision processes to be both scalable and transferable across domains without loss of interpretability or logical integrity.

Future work may focus on extending the notion of N-soft homeomorphism to dynamic systems where evaluation parameters evolve over time. Additionally, exploring its interaction with topological properties such as compactness, separation axioms, or connectedness may

yield deeper insights into the stability and behavior of decision systems under transformation. Real-world applications in areas such as healthcare prioritization, smart city planning, and intelligent recommendation systems also represent promising directions for further exploration.

## Conflict of Interest Statement

There is no conflict of interest between the authors.

## Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

## Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

## Contributions of the Authors

N.G.: Conceptualization, methodology, data processing and analysis, manuscript writing, overall supervision.

C.S.: Theoretical development, preparation of examples and tables, result evaluation, manuscript editing.

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